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Original Article

On Hochschild Cohomology of Generalized Matrix Algebras

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Abstract

Let $S = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ be an algebraover an arbitrary commutative ring, where *A* and *B* are two *R*-algebras with units 1_A and 1_B, respectively, *M* is a left *A*-module and right *B*-module ((*A* - *B*)-module), *N* is a left *B*-module and right *A*-module ((*B* - *A*)-module). We call this algebra, generalized matrix algebra. For

$$X_n^A = A^{n+1} \otimes_R (A \oplus M), \quad X_n^B = (B \oplus M) \otimes_R B^{n+1}$$

and

$$Y_n^A = (N \oplus A) \otimes_R A^{n+1}, \quad Y_n^B = B^{n+1} \otimes_R (n \oplus B)$$

we show that the complex $M \otimes_B N \xleftarrow{\mu} \frac{X_1}{X_1^A \oplus Y_1^A \oplus X_1^B \oplus Y_1^B} \xleftarrow{b'_2} \frac{X_2}{X_2^A \oplus Y_2^A \oplus X_2^B \oplus Y_2^B} \xleftarrow{b'_3} \dots$ is a relative projective resolution of $M \otimes_B N$ as an *S*-module. For an *S*-module *X*, we prove that $\text{üüäm}_{A^e}\left(\left(A^{*+2}\ b'_*\right)\ X_{AA}\right) \cong\ om_{S^e}\left(\left(X_*^A \oplus Y_*^A\ b'_*\right)\ X\right)$ and $\text{üüäm}_{B^e}\left(\left(B^{*+2}\ b_*\right)\ X_{BB}\right) \cong\ om_{S^e}\left(\left(X_*^B \oplus Y_*^B\ b'_*\right)\ X\right)$ where A^e is the enveloping algebra of *A* and $X_{AA} = 1_A X 1_A$, $X_{AB} = 1_A X 1_B$, $X_{BA} = 1_B X 1_A$ and $X_{BB} = 1_B X 1_B$. Moreover, the existence of two long exact sequences of *R*-modules relating the Hochschild cohomology of *A*, *B*, *M*, *N* and *S*are considered and investigated.

Key words: Algebra, Generalized matrix algebra, Hochschild cohomology, A projective resolution

INTRODUCTION

Let *R* be an arbitrary commutative ring with unit, let *A* and *B* be two *R*-algebras with unit, let *M* be a left *A*-module and right *B*-module ((*A* - *B*)-module), *N* be a left *B*-module and right *A*-module ((*B* - *A*)-module) and consider $S = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ as an algebra over *R* with the following operations:

$$\begin{bmatrix} a_1 & m_1 \\ n_1 & b_1 \end{bmatrix} + \begin{bmatrix} a_2 & m_2 \\ n_2 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & m_1 + m_2 \\ n_1 + n_2 & b_1 + b_2 \end{bmatrix}$$

and

$$\begin{bmatrix} a_1 & m_1 \\ n_1 & b_1 \end{bmatrix} . \begin{bmatrix} a_2 & m_2 \\ n_2 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 a_2 + m_1 \otimes_B n_2 & a_1 m_2 + m_1 b_2 \\ n_1 a_2 + b_1 n_2 & n_1 \otimes_A m_2 + b_1 b_2 \end{bmatrix}$$

where $a_1, a_2 \in A$, $b_1, b_2 \in B$, $m_1, m_2 \in M$, $n_1, n_2 \in N$. This algebra is called generalized matrix algebra. In this paper, we suppose that *A* - *B*-module *M* and *B* - *A*-module *N* are *R*-modules.

Also, in *S* if $N = 0$ then *S* is called triangular algebra, and this algebra studied by Cheung [3, 4], and other researchers in [5, 6, 7].

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Let 1_A and 1_B denote the unit elements of A and B. Let X be an A and B two sided module, we write $X_{AA} = 1_A X 1_A$, $X_{AB} = 1_A X 1_B$, $X_{BA} = 1_B X 1_A$ and $X_{BB} = 1_B X 1_B$ [11].

Suppose that A is an algebra and X is an A-module (i.e. X is a two sided A-module). Let $C^n(A,X)$ be the space of all n-linear (as a A-module map) mappings from $A \times ... \times A$(n times) into X and $C^0(A,X) = X$, for $n = 0,1,2, ...$

Consider the sequance

$$0 \to C^0(A,X) \overset{d^0}{\to} C^1(A,X) \overset{d^1}{\to} \dots \overset{d^n}{\to} C^{n+1}(A,X) \to \dots (\tilde{C}(A,X))$$

in which

$$d^0 x(a) = ax - xa \tag{1}$$

$$d^n f(a_1, a_2, ..., a_{n+1}) = a_1 f(a_2, ..., a_{n+1}) + (-1)^{n+1} f(a_1, ..., a_n, a_{n+1})$$

$$+\sum_{j=1}^{n} (-1)^j f(a_1, \dots, a_{j-1}, a_j a_{j+1}, \dots, a_{n+1}) \tag{2}$$

where $n \geq 1$, $x \in X$ and $a_1, ..., a_{n+1} \in A$. The above sequence is a complex for A and X. The n-th cohomology group of \tilde $\widetilde{C}(A,X)$ is said to be n-th Hochschild cohomology group and denoted by $H^n(A,X)$. Actually, $H^n(A,X) = Z^n(A,X)/B^n(A,X)$ where $Z^n(A,X) = \text{Ker } d^n$ and $B^n(A,X) = \text{I}md^{n-1}$, for more details [1, 10].

For a given R-algebra Λ we will denote by Λ^e the enveloping algebra of Λ, that is $\Lambda e = \Lambda \otimes_R \Lambda^{op}$, ($\Lambda^{op}$ means opposite ring Λ), and for $\Lambda \in \Lambda$, Λ^0 will be Λ considered as an element in Λ^{op}.

One method for defining the Hochschild cohomology groups of A is to consider the enveloping algebra A^e of A. For any A-module X we can see it as a left A^e-module by the action$(a \otimes b^0)x = axb$ for every $a,b \in A$, and $x \in X$. Clearly, A is a left A^e-module.

Now, it is easy to verify that the mapping $M \to \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ which maps m into $\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ is a S, S^e, $S \otimes_R B^{op}$, $B \otimes_R A^{op}$-isomorphism. If e is an idempotent element $\begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}$ of S then we denote by P_e the indecomposable projective S-module S_e. Similarly the S^e-module P_e may be regarded as a $S \otimes A^{op}$-module via π, since $P_e(\text{Ker } \pi) = 0$, and finally, the morphism π makes A a S^e-module.

According to the above, the canonical sequence

$$0 \to M \to P_e \to R \to 0$$

is S, S^e, and $S \otimes A^{op}$-exact.

In addition, the mapping $\text{H}om_{\Lambda^{op}}(A,X) \to X$ which sends any A^{op}-morphism $f: R \to X$ into the element $f(1_A)$ of X is a S^e-isomorphism. Using now the fact that the canonical sequence

$$0 \to P_e \to S \to B \to 0 \tag{3}$$

is S^e-exact we obtain the following.

Theorem 1.1 *Let* $S = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$. *Then*

$\text{H}om_{S^e}(S,B) \cong \text{H}om_{S^e}(B,B) \cong \text{H}om_{B^e}(B,B) = \text{H}^0(B,B)$.

Proof. The sequence (3) implies the long exact sequence

$$0 \to \text{H}om_{S^e}(B,B) \to \text{H}om_{S^e}(S,B) \to \text{H}om_{S^e}(P_e,B)$$

$$\to \text{E}xt^1_{S^e}(B,B) \to \text{E}xt^1_{S^e}(S,B) \to \text{E}xt^1_{S^e}(P_e,B) \to ...$$

It follows from the structure of S^e-module of B that $\text{H}om_{S^e}(B,B) \cong \text{H}om_{B^e}(B,B) = ^{H0}(A,A)$.

MAIN RESULTS

Throughout this section, R, A, B, M and N are the defined ring, algebras and modules that are defined in the previous section. Smilar to [7], we denote the canonical resolution of S by (S^{*+2}, b'_*) and assume that (X_*, b'_*) is a S-module subcomplex of (S^{*+2}, b'_*), defined by

$$X_n = A^{n+2} \oplus B^{n+2} \oplus \bigoplus_{i=0}^{n+1} A^i \otimes_R M \otimes_R B^{n+1-i} \oplus \bigoplus_{i=0}^{n+1} B^i \otimes_R N \otimes_R A^{n+1-i}.$$

By a simple calculation one can show that (S^{*+2}, b'_*) is a direct summand of (S^{*+2}, b'_*) as an S-module complex. Hence, (X_*, b'_*) is a projective resolution of the S^e-module S, relative to the family of the S^e-linear epimorphisms which split as R-linear morphisms.

Now, let (X_*^A, b'_*), (X_*^B, b'_*), (Y_*^A, b'_*) and (Y_*^B, b'_*) be the subcomplexes of (X_*, b'_*), defined by

X_n^A üüd$^{n+1} \otimes_R A \oplus M$ X_n^B $B \oplus M \otimes_R B^{n+1}$

and

$$Y_n^A = (N \oplus A) \otimes_R A^{n+1}, \quad Y_n^B = B^{n+1} \otimes_R (n \oplus B).$$

Then projective resolutions of the S^e-modules $1_A S$, $S 1_B$, $S 1_A$ and $1_B S$ are (X_*^A, b'_*), (X_*^B, b'_*), (Y_*^A, b'_*) and (Y_*^B, b'_*), respectively. The proof of the following Lemma is clear and we omit it.

Lemma 2.1 *Let X_i' s be as above for $n \geq 1$ and let*

$$\mu : \frac{X_1}{X_1^A \oplus Y_1^A \oplus X_1^B \oplus Y_1^B} \to M \otimes_B N$$ *be an R-map defined by*

$\mu\,((a \otimes m \otimes b) \oplus (b' \otimes n \otimes a')) = aa'(m \otimes n)bb'$,

for $a,a' \in A$, $b,b' \in B$, $m \in M$ and $n \in N$. The complex

$$M \otimes_B N \xleftarrow{\mu} \frac{X_1}{X_1^A \oplus Y_1^A \oplus X_1^B \oplus Y_1^B} \xleftarrow{b_2'} \frac{X_2}{X_2^A \oplus Y_2^A \oplus X_2^B \oplus Y_2^B} \xleftarrow{b_3'} \cdots \tag{4}$$

is a relative projective resolution of $M \otimes_B N$ as an S-module. A contracting homotopy of (4) as a complex of R-modules is the family

$$\sigma_1 : M \otimes_B N \to \frac{X_1}{X_1^A \oplus Y_1^A \oplus X_1^B \oplus Y_1^B} \tag{5}$$

and

$$\sigma_{n+1} : \frac{X_n}{X_n^A \oplus Y_n^A \oplus X_n^B \oplus Y_n^B} \to \frac{X_{n+1}}{X_{n+1}^A \oplus Y_{n+1}^A \oplus X_{n+1}^B \oplus Y_{n+1}^B} \quad (n \geq 1) \tag{6}$$

defined by

$\sigma_1(m \otimes n) = 1_A \otimes m \otimes 1_B + 1_B \otimes n \otimes 1_A$

$\sigma_{n+1}(a_0 \otimes m \otimes b_{2,n+1} + b_0 \otimes n \otimes a_{2,n+1}) = 1_A \otimes a_0 \otimes m \otimes b_{2,n+1}$

$+(-1)^n 1_A \otimes a_0 m \otimes b_{2,n+1} \otimes 1_B + (-1)^n 1_B \otimes b_0 n \otimes a_{2,n+1} \otimes 1_A$

$\sigma_{n+1}(a_{0,i} \otimes m \otimes b_{i+2,n+1}) = 1_A \otimes a_{0,i} \otimes m \otimes b_{i+2,n+1}$

$+1_B \otimes b_{0,i} \otimes n \otimes a_{i+2,n+1}$ *for* $i > 0$,

where $a_{0,i} = a_0 \otimes \ldots \otimes a_i$ and $b_{i+2,n+1} = b_{i+2} \otimes \ldots \otimes b_{n+1}$.

Let X be an A-module, then for every $f \in \mathrm{Hom}_{A^e}(A^{n+2},X)$,

$$f(a_0 \otimes \ldots \otimes a_{n+1}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} f(a_0 \otimes \ldots \otimes a_{n+1})$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in X_{AA}. \tag{7}$$

for every $a_i \in A$ and $n_0 \leq n \leq n+1$. Similar to (7) we have for every $f \in \mathrm{Hom}_{B^e}(B^{n+2},X)$. As a generalization of Lemma 4 of [7], we have the following result:

Lemma 2.2 Let X be an S-module, then

$$\mathrm{Hom}_{A^e}\left(\left(A^{*+2},b_*'\right),X_{AA}\right) \cong \mathrm{Hom}_{S^e}\left(\left(X_*^A \oplus Y_*^A,b_*'\right),X\right), \tag{8}$$

and

$$\mathrm{Hom}_{B^e}\left(\left(B^{*+2},b_*\right),X_{BB}\right) \cong \mathrm{Hom}_{S^e}\left(\left(X_*^B \oplus Y_*^B,b_*'\right),X\right), \tag{9}$$

Proof. In light of (7), the canonical inclusion i_n: $\mathrm{Hom}_{A^e}(A^{n+2},X^A) \to \mathrm{Hom}_{A^e}(A^{n+2},X)$ is an isomorphism. Let $\theta_n^A : \mathrm{Hom}_{A^e}\left(A^{n+2},X\right) \to \mathrm{Hom}_{S^e}\left(A_n^A \oplus Y_n^A,X\right)$ be an R-map defined by

- $\theta_n^A(f)(a_0 \otimes \ldots \otimes a_{n+1}) = f(a_0 \otimes \ldots \otimes a_{n+1})$,
- $\theta_n^A(f)(a_0 \otimes \ldots \otimes a_n \otimes m) = f(a_0 \otimes \ldots \otimes a_n \otimes 1_A)\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$,
- $\theta_n^A(f)(t \otimes a_0 \otimes \ldots \otimes a_n) = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} f(1_A \otimes a_0 \otimes \ldots \otimes a_n)$,

for every $a_i \in A$, $m \in M$, $t \in N$. Let υ_n^A: $\mathrm{Hom}_{S^e}(X_n^A \oplus Y_n^A,X) \to \mathrm{Hom}_{A^e}(A^{n+2}, X)$ be an R-map defined by restriction. Clearly, $\upsilon_n^A \circ \theta_n^A = id$. Therefore, we shall show that $\theta_n^A \circ \upsilon_n^A = id$. Let $\varphi \in \mathrm{Hom}_{S^e}\left(X_n^A \oplus Y_n^A,X\right)$. Clearly,

$$\theta_n^A \circ \upsilon_n^A(\phi)(a_0 \otimes \ldots \otimes a_{n+1}) = \phi(a_0 \otimes \ldots \otimes a_{n+1})$$

for all $a_0, \ldots, a_{n+1} \in A$. Then

$$\phi(a_0 \otimes \ldots \otimes a_n \otimes m) = \phi(a_0 \otimes \ldots \otimes a_n \otimes 1_A)\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$$

$$= \theta_n^A(\upsilon_n^A(\phi))(a_0 \otimes \ldots \otimes a_n \otimes 1_A)\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$$

$$= \theta_n^A(\upsilon_n^A(\phi))(a_0 \otimes \ldots \otimes a_n \otimes m),$$

and

$$\phi(n \otimes a_0 \otimes \ldots \otimes a_n) = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}\phi(1_A \otimes a_0 \otimes \ldots \otimes a_n)$$

$$= \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}\theta_n^A(\upsilon_n^A(\phi))(1_A \otimes a_0 \otimes \ldots \otimes a_n)$$

$$= \theta_n^A(\upsilon_n^A(\phi))(n \otimes a_0 \otimes \ldots \otimes a_n),$$

for all $a_0, \ldots, a_n \in A$, $m \in M$ and $n \in N$. Hence, $\theta_n^A \circ \upsilon_n^A(\phi) = \phi$. Since the family $\theta_* \circ \upsilon_*$ is an R-map of complexes, (8) holds. The proof of (9) is similar.

Let X be an A and B-module, then for every $f \in \mathrm{Hom}_{S^e}\left(\frac{X_n}{X_n^A \oplus Y_n^A \oplus X_n^B \oplus Y_n^B}, X\right)$, we have

$$f(x_0 \otimes \ldots \otimes x_{n+1}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} f(x_0 \otimes \ldots \otimes x_{n+1})$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in X_{AB}. \tag{10}$$

Lemma 2.3 Let X be a S-module, then

$$\mathrm{Hom}_{A\otimes_R B^{op}}\left(\left(\frac{X_*}{X_*^A \oplus Y_*^A \oplus X_*^B \oplus Y_*^B \oplus}, b_*'\right), X_{AB}\right) \cong$$

$$\mathrm{Hom}_{S^e}\left(\left(\frac{X_*}{X_*^A \oplus Y_*^A \oplus X_*^B \oplus Y_*^B \oplus}, b_*'\right), X\right).$$

Proof. The relation (10) implies that the canonical inclusion

$$\mathrm{Hom}_{S^e}\left(\frac{X_n}{X_n^A \oplus Y_n^A \oplus X_n^B \oplus Y_n^B}, X_{AB}\right) \rightarrow$$

$$\mathrm{Hom}_{S^e}\left(\frac{X_n}{X_n^A \oplus Y_n^A \oplus X_n^B \oplus Y_n^B}, X\right) \tag{11}$$

is an isomorphism. Definition of S^e and (11) complete the proof.

We are now ready to provide one of our main theorems.

Theorem 2.4 Let X be an S-module, then there exists a long exact sequence

$$0 \rightarrow H^0(S,X) \xrightarrow{j} H^0(A, X_{AA}) \oplus H^0(B, X_{BB}) \xrightarrow{\delta^0} \mathrm{Ext}^0_{A\otimes B^{op},R}(M \otimes_B N, X_{AB})$$

$$\xrightarrow{\pi^0} H^1(S,X) \xrightarrow{j^1} H^1(A, X_{AA}) \oplus H^1(B, X_{BB}) \xrightarrow{\delta^1} \mathrm{Ext}^1_{A\otimes B^{op},R}(M \otimes_B N, X_{AB}) \rightarrow \ldots$$

where $\mathrm{Ext}^*_{A\otimes B^{op},R}(M \otimes N, X_{AB})$ denote the E*xt* groups of the $A \otimes_B B^{op}$-module $M \otimes_B N$, relative to the family of the $A \otimes_B B^{op}$-epimorphisms which splits as R-morphisms.

Proof. Let (S^{*+2}, b_*') be the canonical resolution of S and (X_*, b_*'), (X_*^A, b'_*), (X_*^b, b'_*), (Y_*^A, b'_*), (Y_*^B, b'_*) be as before. By Lemma 2.1,

$$0 \rightarrow (X_*^A, b_*') \oplus (Y_*^A, b_*') \oplus (X_*^B, b_*') \oplus (Y_*^B, b_*')$$

$$\rightarrow (X_*, b_*') \rightarrow \left(\frac{X_*}{X_*^A \oplus Y_*^A \oplus X_*^B \oplus Y_*^B}, b_*'\right) \rightarrow 0,$$

implies the long exact sequence

$$0 \rightarrow \mathrm{Ext}^0_{S^e,R}(S,X) \rightarrow \mathrm{Ext}^0_{S^e,R}(1_A S \oplus S1_B, X)$$

$$\rightarrow \mathrm{Ext}^0_{S^e,R}(M \otimes_B N, X) \rightarrow \mathrm{Ext}^1_{S^e,R}(S,X)$$

$$\rightarrow \mathrm{Ext}^1_{S^e,R}(1_A \oplus S1_B, X) \rightarrow \mathrm{Ext}^1_{S^e,R}(M \otimes_B N, X) \rightarrow \ldots.$$

Now, it suffices to apply Lemmas 2.2 and 2.3.

Let $\pi: S \rightarrow B$ be an ring morphism defined by

$$\pi\left(\begin{pmatrix} a & m \\ n & b \end{pmatrix}\right) = b,$$

for all $\begin{pmatrix} a & m \\ n & b \end{pmatrix} \in S$. Let B_S denote the ring B considered as an S-module via π. Also, $(X_{\mathring{a}}^A, b_{\mathring{a}}')$, $(X_{\mathring{a}}^B, b_{\mathring{a}}')$, $(Y_{\mathring{a}}^A, b_{\mathring{a}}')$ and $(Y_{\mathring{a}}^B, b_{\mathring{a}}')$ are projective resolutions. Now, by this notifications we have the next result, that has an important role for proving Theorem 2.7 and it's proof is obvious.

Lemma 2.5 Let $\mu: \frac{X_0}{X_0^A \oplus Y_0^A} \rightarrow B_S$ be the map defined by

$$\mu(b_0 \otimes b_1 + m \otimes b + n \otimes a) = b_0 b_1,$$

for $b, b_0, b_1 \in B$, $a \in A$, $m \in M$ and $n \in N$. The complex

$$B_S \xleftarrow{\mu} \frac{X_0}{X_0^A \oplus Y_0^A} \xleftarrow{b_1'} \frac{X_1}{X_1^A \oplus Y_1^A} \xleftarrow{b_2'} \frac{X_2}{X_2^A \oplus Y_2^A} \xleftarrow{b_3'} \ldots \tag{12}$$

is a relative projective resolution of B_S as an S-module. A contracting homotopy of (12) as a complex of R-modules is the family

$$\sigma_0: B_S \rightarrow \frac{X_0}{X_0^A \oplus Y_0^A} \ \textit{and} \ \sigma_{n+1}: \frac{X_n}{X_n^A \oplus Y_n^A} \rightarrow \frac{X_{n+1}}{X_{n+1}^A \oplus Y_{n+1}^A} \ (n \geq 0),$$

defined by:

$$\sigma_{n+1}(x_0 \otimes \ldots \otimes x_n) = 1_A \otimes x_0 \otimes \ldots \otimes x_n.$$

Lemma 2.6 Let X be an S-module, then

$$\mathrm{Hom}_{S\otimes_R B^{op}}\left(\left(\frac{X_*}{X_*^A\oplus Y_*^A},b'_*\right),X1_B\right)\cong$$
$$\mathrm{Hom}_{S^e}\left(\left(\frac{X_*}{X_*^A\oplus Y_*^A},b'_*\right),X\right).$$

Proof. Since, for every $f\in \mathrm{Hom}_{S^e}\left(\frac{X_n}{X_n^A\oplus Y_n^A},X\right)$,

$$\ddot{u}\ddot{u}\bar{\ddot{u}}\bar{\ddot{u}}\ddot{u}_0\otimes\ldots\otimes \quad {}_{n+1} \qquad {}_0\otimes\ldots\otimes \quad {}_{n+1}\begin{pmatrix}0&0\\0&1\end{pmatrix}\in \quad {}_B$$

the canonical inclusion

$$\mathrm{Hom}_{S^e}\left(\frac{X_n}{X_n^A\oplus Y_n^A},X1_B\right)\to \mathrm{Hom}_{S^e}\left(\frac{X_n}{X_n^A\oplus Y_n^A},X\right)$$

is an isomorphism. To end the proof it suffices to observe that

$$\mathrm{Hom}_{S\otimes_R B^{op}}\left(\frac{X_n}{X_n^A\oplus Y_n^A},X1_B\right)\cong$$
$$\mathrm{Hom}_{S^e}\left(\frac{X_n}{X_n^A\oplus Y_n^A},X1_B\right).$$

By the above lemmas, we are ready to prove the following.

Theorem 2.7 Let X be an S-module, then there exists a long exact sequence

$$0\to \mathrm{Ext}^0_{S\otimes_R B^{op},R}(B_S,X1_B)\to H^0(S,X)\to H^0(A,X_{AA})$$

$$\to \mathrm{Ext}^1_{S\otimes_R B^{op},R}(B_S,X1_B)\to H^1(S,X)\to H^1(A,X_{AA})\to\ldots.$$

Proof. By Lemma 2.5,

$$0\to (X_*^A\oplus Y_*^A,b'_*)\to (X_*,b'_*)\to\left(\frac{X_*}{X_*^A\oplus Y_*^A},b'_*\right)\to 0,$$

leads to the long exact sequance

$$0\to \mathrm{Ext}^0_{S^e,R}(B_S,X)\to \mathrm{Ext}^0_{S^e,R}(S,X)\to \mathrm{Ext}^0_{S^e,R}(1_AS,X)$$

$$\to \mathrm{Ext}^1_{S^e,R}(B_S,X)\to \mathrm{Ext}^1_{S^e,R}(S,X)\to \mathrm{Ext}^1_{S^e,R}(1_AS,X)\to\ldots.$$

Now, apply Lemmas 2.2 and 2.6.

Example 2.8 Let $S=\begin{bmatrix}A&M\\N&B\end{bmatrix}$ where $M\otimes_B N=0=N\otimes_B M$. Now, we compare $H^n(S,X)$ where X is an S-module. Since, $M\otimes N=0$, thus $\mathrm{Ext}^i_{A\otimes B^{op},R}(M\otimes N,X_{AB})=0$.

Therefore, by Theorem 2.4, we have

$$H^n(S,X)\cong H^n(A,X_{AA})\oplus H^n(B,X_{BB}).$$

Also,

$$H^n(S,S)\cong H^n(A,A)\oplus H^n(B,B).$$

Example 2.9 In example 2.8, we put $M=\mathbb{Z}_m$ *and* $N=\mathbb{Z}_n$ where (m,n) = 1 *and* $A=B=\mathbb{Z}$. *Then*

$$H^n(S,S)\cong H^n(\mathbb{Z},\mathbb{Z})\oplus H^n(\mathbb{Z},\mathbb{Z})=0.$$

REFERENCES

[1] Brodmann MP and Sharp RY (1998) Local cohomology: an algebraic introduction with geometric applications, Cambridge University Press.

[2] Du Y., Wang Y. Lie derivations of generalized matrix algebras. Linear Alg. Appl.,2012, 437, 2719-2726.

[3] Cheung W. S. Commuting maps of triangular algebras. J. London Math. Soc., 2001, 63(1), 117-127.

[4] Cheung W. S. Lie derivations of triangular algebras. Linear Multilinear Alg., 2003, 51(3), 299-310.

[5] Christensen E. Derivations of nest algebras. Math. Ann., 1977, 229,155-161.

[6] Coelho S. P., Milies C. P. Derivations of upper triangular matrix rings. Linear Alg. Appl., 1993, 187, 263-267.

[7] Guccione J. A., Guccione J. J. Hochschild cohomology of triangular matrix algebras.http://arxiv.org/abs/math/0104068v2.

[8] Huneke C. Problems on Local cohomology, in: Free Resolution in Commutative Alg. and Alg. Geometry (Sundance, Utah, 1990) Mathematics, 2, Jones and Barlett,pp.93-108. Research Notes in Mathematics, 2, Jones and Barlett Publisher, Boston, MA, (1992).

[9] Ji P., Qi W. Characterizations of Lie derivations of triangular algebras. Linear Alg. Appl., 2011, 435, 1137-1146.

[10] Rotman J. An introduction to homological algebra. Second Ed., New York, Speringer-verlag, 2009.

[11] Xiao Z. K., Wei F. Commuting mappings of generalized matrix algebras, Linear Alg. Appl., 2010, 433, 2178-2197.

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